TRIGONOMETRY

8

CONTENTS

- Right Angle Triangle
- Trigonometric Ratio (T.R.) of some Specific Angles
- Trigonometric Ratios of Complementary Angles
- Trigonometric Identities

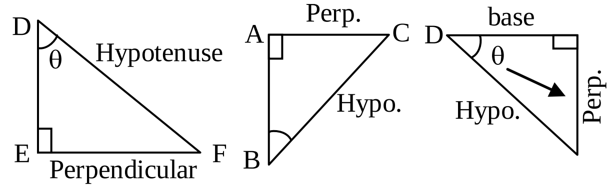
Trigonometry is the branch of mathematics in which we study of relationships between the sides & angles of a triangle.

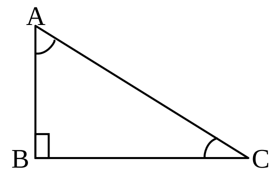
Fact : In Greek words :
Tri = three
gon = sides
metron = measure

The ratio of sides of a right angle triangle with respect to acute angles are called "Trigonometric ratios of the angle".

➢ RIGHT ANGLE TRIANGLE

1. A Δ having one angle equal to 90º is called right angle Δ.
2. The sum of other two acute (Less than 90º) angles is 90º. (or both acute angles are complementary)
3. The side opposite to 90º, is called hypotenuse, it is longest side in Δ.
4. The side opposite to given one acute angle is perpendicular.
5. The rest (IIIrd) side is base.



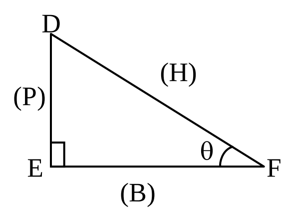


	Hypotenuse	Perpendicular	Base
for $\angle A$	AC	BC	AB
for $\angle C$	AC	AB	BC

The trigonometry ratio are

sine of $\angle\theta$, cosine of $\angle\theta$, tangent of $\angle\theta$, cotangent of $\angle\theta$, secant of $\angle\theta$, cosecant of $\angle\theta$.

These ratios are abbreviated as sin θ, cos θ, tan θ, cot θ, sec θ, cosec θ and the relation with sides are



sin θ	= P/H = DE/DF
cos θ	= B/H = EF/DF
tan θ	= P/B = DE/EF
cot θ	= B/P = EF/DE
sec θ	= H/B = DF/EF
cosec θ	= H/P = DF/DE

By above table $\sin\theta = \dfrac{1}{\text{cosec}\theta}$, $\cos\theta = \dfrac{1}{\sec\theta}$,

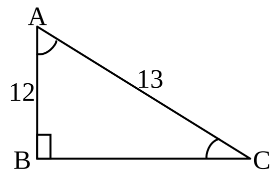
$\tan\theta = \dfrac{1}{\cot\theta}$ also $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{P/H}{B/H} = \dfrac{P}{B}$

$\therefore$ we can say ''Trigonometric Ratio'' represents ratio between acute angles & sides of triangle.

❖ EXAMPLES ❖

Ex.1 If ABC is right angle triangle, $\angle B = 90^\circ$, AB = 12 cm, AC = 13 cm then find sin A and cos C.

Sol. Using Pythagoras theorem



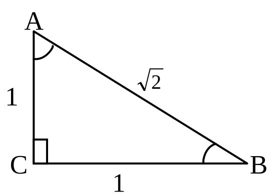
$$BC = \sqrt{AC^2 - AB^2} = \sqrt{169 - 144} = 5 \text{ cm}$$

$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{5}{13}$$

$$\cos C = \frac{AB}{AC} = \frac{12}{13}$$ **Ans.**

Ex.2 If $\sin A = \frac{1}{\sqrt{2}}$ in right triangle ABC, then find value of tan A, cosec A, tan B, cosec B.

Sol.



$$\because \sin A = \frac{1}{\sqrt{2}} = \frac{BC}{AB}$$

$$\therefore AC = \sqrt{AB^2 - BC^2} = \sqrt{(\sqrt{2}k)^2 - (k)^2}$$

$$= \sqrt{2k^2 - k^2} = \sqrt{k^2} = k$$

$$\therefore \tan A = \frac{BC}{AC} = \frac{k}{k} = 1$$

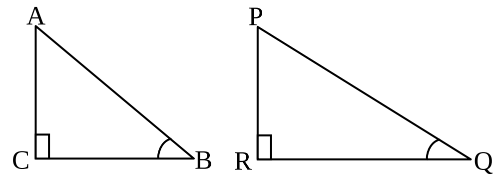
$$\text{cosec } A = \frac{1}{\sin A} = \frac{\sqrt{2}k}{k} = \sqrt{2}$$

$$\tan B = \frac{AC}{BC} = \frac{k}{k} = 1$$

$$\text{cosec } B = \frac{AB}{AC} = \frac{\sqrt{2}k}{k} = \sqrt{2}$$

Ex.3 If $\angle B$ and $\angle Q$ are acute angles such that sin B = sin Q, then prove that $\angle B = \angle Q$. **[NCERT]**

Sol. Let us consider two right triangles ABC and PQR where sin B = sin Q.



We have $\sin B = \frac{AC}{AB}$

and $\sin Q = \frac{PR}{PQ}$

Then $\frac{AC}{AB} = \frac{PR}{PQ}$

Therefore, $\frac{AC}{PR} = \frac{AB}{PQ} = k$, say ....(1)

Now, using Pythagoras theorem,

$$BC = \sqrt{AB^2 - AC^2}$$

and $QR = \sqrt{PQ^2 - PR^2}$

So, $$\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}}$$

$$= \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}}$$

$$= \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \quad .....(2)$$

From (1) and (2), we have

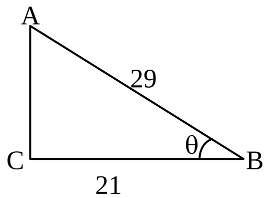
$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

Then, by using Theorem, $\Delta ACB \sim \Delta PRQ$ and therefore, $\angle B = \angle Q$.

Ex.4 Consider ΔACB, right-angled at C, in which AB = 29 units, BC = 21 units and $\angle ABC = \theta$ (see figure). Determine the value of

(i) $\cos^2 \theta + \sin^2\theta$,

(ii) $\cos^2 \theta - \sin^2\theta$ **[NCERT]**



Sol. In ΔACB, we have

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$$
$$= \sqrt{(29-21)(29+21)} = \sqrt{(8)(50)}$$
$$= \sqrt{400} = 20 \text{ units}$$

So, $\sin\theta = \frac{AC}{AB} = \frac{20}{29}$, $\cos\theta = \frac{BC}{AB} = \frac{21}{29}$

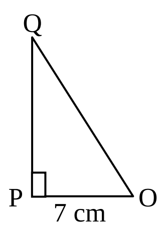
Now, (i) $\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2$

$$= \frac{20^2 + 21^2}{29^2} = \frac{400 + 441}{841} = 1,$$

and (ii) $\cos^2\theta - \sin^2\theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2$

$$= \frac{(21+20)(21-20)}{29^2} = \frac{41}{841}.$$

Ex.5 In Δ OPQ, right-angled at P, OP = 7 cm and OQ – PQ = 1 cm (see figure). Determine the values of sin Q and cos Q. **[NCERT]**



Sol. In ΔOPQ, we have

$OQ^2 = OP^2 + PQ^2$

i.e $(1 + PQ)^2 = OP^2 + PQ^2$

i.e. $1 + PQ^2 + 2PQ = OP^2 + PQ^2$

i.e. $1 + 2PQ = 7^2$

i.e. PQ = 24 cm and OQ = 1 + PQ = 25 cm

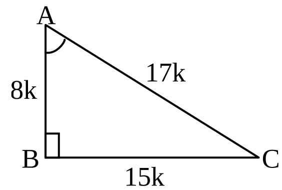
So, $\sin Q = \frac{7}{25}$ and $\cos Q = \frac{24}{25}$

Note :

1. The values of sin θ & cos θ are always less than or equal to 1 & greater than or equal to –1.
2. Value of tan θ & cot θ lie between $-\infty$ to $+\infty$.
3. sin A, cos A, etc. are not product of sin and A.
4. $(\sin A)^2 \neq \sin A^2$ etc.

Ex.6 Given 15 cot A = 8, find sin A and sec A. **[NCERT]**

Sol. $\cot A = \frac{8}{15} = \frac{\text{base}}{\text{perpendicular}}$



$$AC = \sqrt{AB^2 + BC^2} = \sqrt{64k^2 + 225k^2}$$
$$= \sqrt{289k^2} = 17k$$

$$\sin A = \frac{BC}{AC} = \frac{15k}{17k} = \frac{15}{17}$$

$$\sec A = \frac{AC}{AB} = \frac{17k}{8k} = \frac{17}{8}$$ **Ans.**

Ex.7 Given $\sec\theta = \frac{13}{12}$, calculate all other trigonometric ratios. **[NCERT]**

Sol. $\because \sec\theta = \frac{13}{12} = \frac{\text{Hypotenuse}}{\text{Base}}$

$\therefore$ perpendicular $= \sqrt{(13k)^2 - (12k)^2}$

$$= \sqrt{(169-144)k^2} = 5\,k$$

$\therefore \sin\theta = \frac{P}{H} = \frac{5k}{13k} = \frac{5}{13}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{12}{13}$

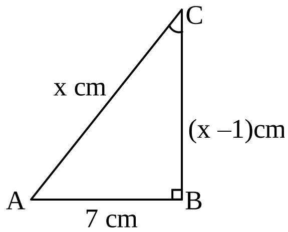
$\tan\theta = \frac{P}{B} = \frac{5k}{12k} = \frac{5}{12}$

$\cot\theta = \frac{B}{P} = \frac{12k}{5k} = \frac{12}{5}$

$\text{cosec}\,\theta = \frac{H}{P} = \frac{13k}{5k} = \frac{13}{5}$

Ex.8 In ΔABC, right –angled at B, AB = 7 cm and (AC – BC) = 1 cm. Find the values of sin C and cos C.

Sol. Consider ΔABC in which $\angle B = 90^\circ$, AB = 7 cm and (AC – BC) = 1 cm.



Let AC = x cm.

Then, BC = (x – 1) cm

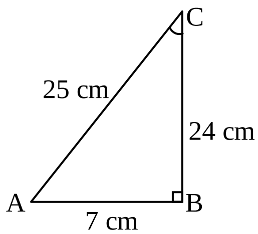
By Pythagoras theorem, we have :

$AB^2 + BC^2 = AC^2 \Rightarrow (7)^2 + (x-1)^2 = x^2$

$\Rightarrow$ $49 + x^2 - 2x + 1 = x^2$

$\Rightarrow$ $2x = 50$

$\Rightarrow$ $x = 25$



$\therefore$ AC = 25 cm, BC = (25 – 1)cm = 24 cm and AB = 7 cm.

For T-ratios of $\angle C$, we have

base = BC = 24 cm,

perpendicular = AB = 7 cm and

hypotenuse = AC = 25 cm.

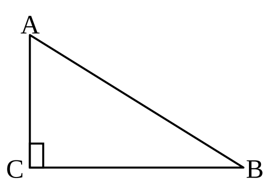
$\therefore \sin C = \dfrac{AB}{AC} = \dfrac{7}{25}$ and $\cos C = \dfrac{BC}{AC} = \dfrac{24}{25}$.

Ex.9 If $\angle A$ and $\angle B$ are acute angles such that $\cos A = \cos B$, then show that $\angle A = \angle B$.

[NCERT]

Sol. $\because \cos A = \cos B$

$\dfrac{AC}{AB} = \dfrac{BC}{AB}$



$\therefore$ AC = BC

$\therefore$ Δ is an isosceles Δ

$\therefore$ $\angle A = \angle B$ Proved.

Ex.10 If $\cot\theta = \dfrac{7}{8}$, evaluate : **[NCERT]**

(i) $\dfrac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$, (ii) $\cot^2\theta$

Sol. $\because \cot\theta = \dfrac{7}{8} = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{B}{P}$

$\therefore H = \sqrt{(8k)^2 + (7k)^2} = \sqrt{(64+49)}k$

$= \sqrt{113}\,k$

$\therefore \sin\theta = \dfrac{P}{H} = \dfrac{8k}{\sqrt{113}k} = \dfrac{8}{\sqrt{113}}$

$\cos\theta = \dfrac{B}{H} = \dfrac{7k}{\sqrt{113}k} = \dfrac{7}{\sqrt{113}}$

(i) $\dfrac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$

$= \dfrac{\left(1+\dfrac{8}{\sqrt{113}}\right)\left(1-\dfrac{8}{\sqrt{113}}\right)}{\left(1+\dfrac{7}{\sqrt{113}}\right)\left(1-\dfrac{7}{\sqrt{113}}\right)}$

$= \dfrac{(\sqrt{113}+8)(\sqrt{113}-8)}{(\sqrt{113}+7)(\sqrt{113}-7)}$

$= \dfrac{113-64}{113-49} = \dfrac{49}{64}$ **Ans.**

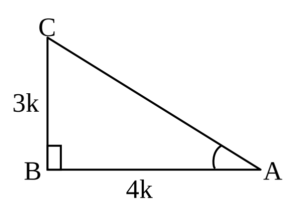
(ii) $\cot^2\theta = \left(\dfrac{B}{P}\right)^2 = \left(\dfrac{7k}{8k}\right)^2 = \dfrac{49}{64}$ **Ans.**

Ex.11 If 3 cot A = 4, check whether

$\dfrac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2 A$ or not.

[NCERT]

Sol. $\because \cot A = \dfrac{4}{3}$ $\therefore \tan A = \dfrac{3}{4}$



$\therefore$ $AC = \sqrt{AB^2 + BC^2} = \sqrt{16k^2 + 9k^2}$

$= \sqrt{25k^2} = 5k$

$\therefore \sin A = \dfrac{3k}{5k} = \dfrac{3}{5}$

$\cos A = \frac{4k}{5k} = \frac{4}{5}$

LHS = $\frac{1 - \tan^2 A}{1 + \tan^2 A}$

$= \frac{1 - \left(\frac{3}{4}\right)^2}{1 + \left(\frac{3}{4}\right)^2} = \frac{1 - \frac{9}{16}}{1 + \frac{9}{16}}$

$= \frac{(16-9)/16}{(16+9)/16} = \frac{7}{25}$

RHS = $\cos^2 A - \sin^2 A$

$= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

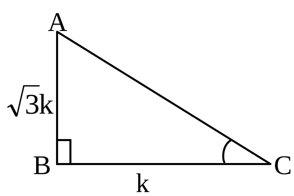
LHS = RHS

Ex.12 In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of : **[NCERT]**

(i) sin A cos C + cos A sin C

(ii) cos A cos C – sin A sin C

Sol. $\tan A = \frac{1}{\sqrt{3}} = \frac{P}{B}$



$\therefore AC = \sqrt{(\sqrt{3}k)^2 + (k)^2} = \sqrt{3k^2 + k^2} = 2k$

$\therefore \sin A = \frac{BC}{AC} = \frac{k}{2k} = \frac{1}{2}$;

$\sin C = \frac{AB}{AC} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$;

$\cos A = \frac{AB}{AC} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$;

$\cos C = \frac{BC}{AC} = \frac{k}{2k} = \frac{1}{2}$

(i) sin A cos C + cos A sin C

$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{4} + \frac{3}{4} = 1$

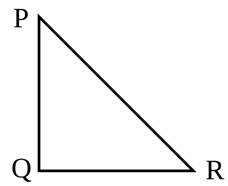
(ii) cos A cos C – sin A sin C

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

Ex.13 In ΔPQR, right-angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P. **[NCERT]**

Sol.



$\because$ PR + QR = 25 cm

PQ = 5 cm

Let PR = x cm

$\therefore$ QR = (25 – x) cm

Using Pythagoras theorem

$PR^2 = PQ^2 + QR^2$

$x^2 = 5^2 + (25 - x)^2$

$x^2 = 25 + 625 + x^2 - 50x$

$\Rightarrow 50x = 650$

$\Rightarrow x = 13$ cm = PR

$\therefore$ QR = 25 – 13 = 12 cm.

$\sin P = \frac{QR}{PR} = \frac{12k}{13k} = \frac{12}{13}$

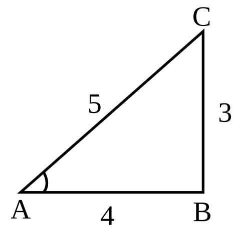
$\cos P = \frac{PQ}{PR} = \frac{5k}{13k} = \frac{5}{13}$

$\tan P = \frac{QR}{PQ} = \frac{12k}{5k} = \frac{12}{5}$ **Ans.**

Ex.14 If $\sin A = \frac{3}{5}$, find cos A and tan A.

Sol. Since $\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}$, so

We draw a triangle ABC, right angled at B such that



Perpendicular = BC = 3 units,

and, Hypotenuse = AC = 5 units.

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow\ 5^2 = AB^2 + 3^2$

$\Rightarrow\ AB^2 = 5^2 - 3^2$

$\Rightarrow\ AB^2 = 16 \quad \Rightarrow \quad AB = 4$

When we consider the t-ratio of $\angle A$, we have Base = AB = 4, Perpendicular = BC = 3, Hypotenuse = AC = 5.

$\therefore\ \cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{4}{5}$

and, $\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{3}{4}$

Ex.15 If cosec A = $\sqrt{10}$, find other five trigonometric ratios.

Sol. Since cosec A = $\dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{\sqrt{10}}{1}$,

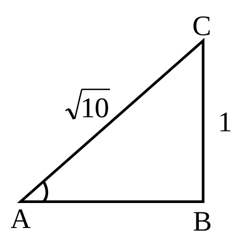
so we draw a right triangle ABC, right angled at B such that

Perpendicular = BC = 1 unit. and,

Hypotenuse = AC = $\sqrt{10}$ units.

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$



$\Rightarrow\ (\sqrt{10})^2 = AB^2 + 1^2$

$\Rightarrow\ AB^2 = 10 - 1 = 9$

$\Rightarrow\ AB = \sqrt{9} = 3$

When we consider the trigonometric ratios of $\angle A$, we have

Base = AB = 3, Perpendicular = BC = 1, and

Hypotenuse = AC = $\sqrt{10}$.

$\therefore\ \sin A = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{1}{\sqrt{10}}$;

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{3}{\sqrt{10}}$;

$\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{1}{3}$;

$\sec A = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{\sqrt{10}}{3}$;

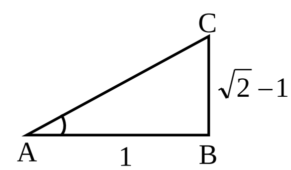
and $\cot A = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{3}{1} = 3$

Ex.16 If $\tan A = \sqrt{2} - 1$, show that $\sin A \cos A = \dfrac{\sqrt{2}}{4}$.

Sol. Since $\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{\sqrt{2}-1}{1}$, so we draw a right triangle ABC, right angled at B such that Base = AB = 1 and Perpendicular = BC = $\sqrt{2} - 1$.

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$



$\Rightarrow\ AC^2 = 1^2 + (\sqrt{2} - 1)^2$

$\Rightarrow\ AC^2 = 1 + 2 + 2 - 2\sqrt{2}$

$\Rightarrow\ AC^2 = 4 - 2\sqrt{2} \ \Rightarrow\ AC = \sqrt{4 - 2\sqrt{2}}$

Now, $\sin A = \dfrac{BC}{AC} = \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}$, and

$\cos A = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

$\therefore\ \sin A \cos A = \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \times \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

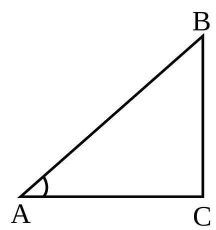
$= \dfrac{\sqrt{2}-1}{4-2\sqrt{2}} = \dfrac{\sqrt{2}-1}{2\sqrt{2}(\sqrt{2}-1)} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$.

- $\sin^2\theta = (\sin\theta)^2$
  $\cos^2\theta = (\cos\theta)^2$
  $\tan^2\theta = (\tan\theta)^2$
  $\text{cosec}^2\theta = (\text{cosec}\,\theta)^2$
  $\sec^2\theta = (\sec\theta)^2$
  $\cot^2\theta = (\cot\theta)^2$

❖ EXAMPLES ❖

Ex.17 In a Δ ABC right angled at C, if $\tan A = \frac{1}{\sqrt{3}}$ and $\tan B = \sqrt{3}$. Show that
$\sin A \cos B + \cos A \sin B = 1$.

Sol. Let us draw a ΔABC, right angled at C in which $\tan B = \sqrt{3}$ and $\tan A = \frac{1}{\sqrt{3}}$.



Now, $\tan A = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{BC}{AC} = \frac{1}{\sqrt{3}} \quad \left[\because \tan A = \frac{BC}{AC}\right]$

$\Rightarrow BC = x$ and $AC = \sqrt{3}x$ ....(i)

And, $\tan B = \sqrt{3}$

$\Rightarrow \frac{AC}{BC} = \frac{\sqrt{3}}{1} \quad \left[\because \tan B = \frac{AC}{BC}\right]$

$\Rightarrow AC = \sqrt{3}\,x$ and $BC = x$ ....(ii)

From (i) and (ii), we have

$BC = x,\ AC = \sqrt{3}\,x$

By Pythagoras theorem, we have

$AB^2 = AC^2 + BC^2$

$\Rightarrow AB^2 = (\sqrt{3}x)^2 + x^2$

$\Rightarrow AB^2 = 3x^2 + x^2$

$\Rightarrow AB^2 = 4x^2$

$\Rightarrow AB = 2x$

When we find the t-rations of ∠A, we have Base = AC = $\sqrt{3}$ x, Perpendicular = BC = x, and Hypotenuse = AB = 2x.

$\therefore \sin A = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2}$ and

$\cos A = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$

When we consider the t-ratios of ∠B, we have Base = BC = x, Perpendicular = AC = $\sqrt{3}$ x, and Hypotenuse = AB = 2x.

$\therefore \cos B = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2}$ and

$\sin B = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$

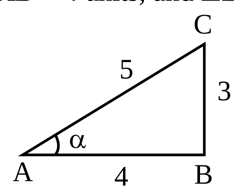
Now,

$\sin A \cos B + \cos A \sin B = \frac{1}{2}\times\frac{1}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}$

$= \frac{1}{4} + \frac{3}{4} = 1.$

Ex.18 If $\sec\alpha = \frac{5}{4}$, evaluate $\frac{1-\tan\alpha}{1+\tan\alpha}$.

Sol. Since $\sec\alpha = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{4}$, so we draw a right triangle ABC, right angled at B such that
Hypotenuse = AC = 5 units,
Base = AB = 4 units, and ∠BAC = α.



By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow 5^2 = 4^2 + BC^2$

$\Rightarrow BC^2 = 5^2 - 4^2 = 9$

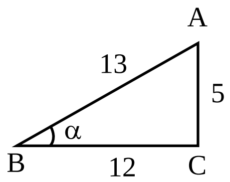
$\Rightarrow BC = \sqrt{9} = 3$

$\therefore \tan\alpha = \frac{BC}{AB} = \frac{3}{4}$

Now, $\frac{1-\tan\alpha}{1+\tan\alpha} = \frac{1-\frac{3}{4}}{1+\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}.$

Ex.19 If $\cot B = \frac{12}{5}$, prove that
$\tan^2 B - \sin^2 B = \sin^4 B.\sec^2 B$.

Sol. Since $\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{12}{5}$, so we draw a right triangle ABC, right angled at C such that Base = BC = 12 units.
Perpendicular = AC = 5 units.



By Pythagoras theorem, we have

$AB^2 = BC^2 + AC^2$

$\Rightarrow$ $AB^2 = 12^2 + 5^2 = 169$

$\Rightarrow$ $AB = \sqrt{169} = 13$

$\therefore$ $\sin B = \dfrac{AC}{AB} = \dfrac{5}{13}$, $\tan B = \dfrac{AC}{BC} = \dfrac{5}{12}$

and $\sec B = \dfrac{AB}{BC} = \dfrac{13}{12}$

Now, LHS = $\tan^2 B - \sin^2 B = (\tan B)^2 - (\sin B)^2$

$= \left(\dfrac{5}{12}\right)^2 - \left(\dfrac{5}{13}\right)^2 = \dfrac{25}{144} - \dfrac{25}{169}$

$= 25\left(\dfrac{1}{144} - \dfrac{1}{169}\right) = 25\left(\dfrac{169-144}{144\times 169}\right)$

$= 25 \times \dfrac{25}{144\times 169} = \dfrac{25\times 25}{144\times 169}$

$= \dfrac{5^2\times 5^2}{12^2\times 13^2}$ ....(i)

and, RHS = $\sin^4 B \sec^2 B$

$= (\sin B)^4 (\sec B)^2$

$= \left(\dfrac{5}{13}\right)^4 \times \left(\dfrac{13}{12}\right)^2$

$= \dfrac{5^4}{13^2\times 12^2}$

$= \dfrac{5^2\times 5^2}{13^2\times 12^2}$ ....(ii)

From (i) and (ii), we have

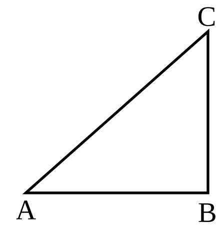
$\tan^2 B - \sin^2 B = \sin^4 B \sec^2 B$.

Ex.20 In a right triangle ABC, right angled at B, the ratio of AB to AC is $1 : \sqrt{2}$. Find the values of

(i) $\dfrac{2\tan A}{1+\tan^2 A}$ and (ii) $\dfrac{2\tan A}{1-\tan^2 A}$

Sol. We have, $AB : AC = 1 : \sqrt{2}$ i.e. $\dfrac{AB}{AC} = \dfrac{1}{\sqrt{2}}$

$\therefore$ AB = x and AC = $\sqrt{2}$ x.



By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow$ $(\sqrt{2}\,x)^2 = x^2 + BC^2$

$\Rightarrow$ $BC^2 = 2x^2 - x^2 = x^2$

$\Rightarrow$ $BC = x$

$\therefore$ $\tan A = \dfrac{BC}{AB} = \dfrac{x}{x} = 1$

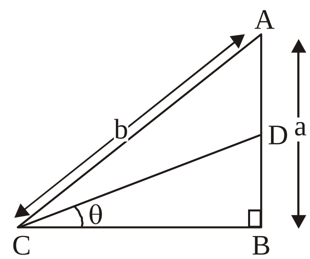
Now, $\dfrac{2\tan A}{1+\tan^2 A} = \dfrac{2\times 1}{1+1^2} = \dfrac{2}{2} = 1$

Now, $\dfrac{2\tan A}{1-\tan^2 A} = \dfrac{2\times 1}{1-1} = \dfrac{2}{0}$, which is undefined.

Ex.21 In fig. AD = DB and $\angle B$ is a right angle. Determine

(i) $\sin\theta$ (ii) $\cos\theta$

(iii) $\tan\theta$ (iv) $\sin^2\theta + \cos^2\theta$



Sol. We have,

$AB = a$

$\Rightarrow$ $AD + DB = a$ [$\because$ AD = DB]

$\Rightarrow$ $AD + AD = a$

$\Rightarrow$ $2AD = a$ $\Rightarrow$ $AD = \dfrac{a}{2}$

Thus, $AD = DB = \dfrac{a}{2}$

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow$ $b^2 = a^2 + BC^2$

$\Rightarrow$ $BC^2 = b^2 - a^2$ $\Rightarrow$ $BC = \sqrt{b^2 - a^2}$

Thus, in ΔBCD, we have

Base = BC = $\sqrt{b^2 - a^2}$

and Perpendicular = BD = $\dfrac{a}{2}$

Applying Pyhthagoras theorem in ΔBCD, we have

$BC^2 + BD^2 = CD^2$

$\Rightarrow \ (\sqrt{b^2 - a^2})^2 + \left(\frac{a}{2}\right)^2 = CD^2$

$\Rightarrow \ CD^2 = b^2 - a^2 + \frac{a^2}{4}$

$\Rightarrow \ CD^2 = \frac{4b^2 - 4a^2 + a^2}{4}$

$\Rightarrow \ CD = \frac{\sqrt{4b^2 - 3a^2}}{2}$

Now,

(i) $\sin\theta = \frac{BD}{CD}$

$\Rightarrow \ \sin\theta = \frac{a/2}{\frac{\sqrt{4b^2 - 3a^2}}{2}} = \frac{a}{\sqrt{4b^2 - 3a^2}}$

(ii) $\cos\theta = \frac{BC}{CD}$

$\Rightarrow \ \cos\theta = \frac{\sqrt{b^2 - a^2}}{\frac{\sqrt{4b^2 - 3a^2}}{2}} = \frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}$

(iii) $\tan\theta = \frac{BD}{CD}$

$\Rightarrow \ \tan\theta = \frac{a/2}{\sqrt{b^2 - a^2}} = \frac{a}{2\sqrt{b^2 - a^2}}$, and

(iv) $\sin^2\theta + \cos^2\theta$

$= \left(\frac{a}{\sqrt{4b^2 - 3a^2}}\right)^2 + \left(\frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}\right)^2$

$= \frac{a^2}{4b^2 - 3a^2} + \frac{4(b^2 - a^2)}{4b^2 - 3a^2}$

$= \frac{4b^2 - 3a^2}{4b^2 - 3a^2} = 1$

➢ TRIGONOMETRIC RATIO (T.R.) OF SOME SPECIFIC ANGLES

The angles 0°, 30°, 45°, 60°, 90° are angles for which we have values of T.R.

$\angle$A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

- $\sin\theta \uparrow$ when $\theta \uparrow$, $0 \le \theta \le 90°$
- $\cos\theta \downarrow$ when $\theta \uparrow$, $0 \le \theta \le 90°$
- $\tan\theta$, $\cot\theta$ are not defined for θ = 90° & 0 respectively.
- cosec θ, sec θ are not defined when θ = 0 & 90° respectively.
- $\sin\theta = \cos\theta$ for only θ = 45°
- $\because \ 180° = \pi^c$
- $\therefore \ 30° = \left(\frac{\pi}{6}\right)^c$; $45° = \left(\frac{\pi}{4}\right)^c$

  $60° = \left(\frac{\pi}{3}\right)^c$; $90° = \left(\frac{\pi}{2}\right)^c$

❖ EXAMPLES ❖

Ex.22 Evaluate each of the following in the simplest form :

(i) sin 60º cos 30º + cos 60º sin 30º

(ii) sin 60º cos 45º + cos 60º sin 45º

Sol. (i) sin 60º cos 30º + cos 60º sin 30º

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$

(ii) sin 60º cos 45º + cos 60º sin 45º

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Ex.23 Evaluate the following expression :

(i) $\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$

(ii) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$.

Sol. (i) $\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$

$\tan 60^\circ (\operatorname{cosec} 45^\circ)^2 + (\sec 60^\circ)^2 \tan 45^\circ$

$= \sqrt{3} \times (\sqrt{2})^2 + (2)^2 \times 1$

$= \sqrt{3} \times 2 + 4 = 4 + 2\sqrt{3}$

(ii) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$

$= 4(\cot 45^\circ)^2 - (\sec 60^\circ)^2 + (\sin 60^\circ)^2 + (\cos 90^\circ)^2$

$= 4 \times (1)^2 - (2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 0$

$= 4 - 4 + \frac{3}{4} + 0 = \frac{3}{4}$

Ex.27 Find the value of θ in each of the following :

(i) $2 \sin 2\theta = \sqrt{3}$ (ii) $2 \cos 3\theta = 1$

Sol.(i) $2 \sin 2\theta = \sqrt{3}$

$\Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin 2\theta = \sin 60^\circ$

$\Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$

(ii) $2 \cos 3\theta = 1$

$\Rightarrow \cos 3\theta = \frac{1}{2}$

$\Rightarrow \cos 3\theta = \cos 60^\circ$

$\Rightarrow 3\theta = 60^\circ \Rightarrow \theta = 20^\circ$.

Ex.31 If $\tan (A + B) = \sqrt{3}$ and $\tan (A - B) = \frac{1}{\sqrt{3}}$;

$0^\circ < A + B \leq 90^\circ$; $A > B$, find A and B.

Sol. $\tan (A + B) = \sqrt{3} = \tan 60^\circ$

& $\tan (A - B) = 1/\sqrt{3} = \tan 30^\circ$

$A + B = 60^\circ$ …….(1)

$A - B = 30^\circ$ …….(2)

$2A = 90^\circ \Rightarrow A = 45^\circ$ **Ans.**

adding (1) & (2)

$A + B = 60$

$A - B = 30$

Sub fact equation (2) from (1)

$A + B = 60$

$A - B = 30$

$-\quad +\quad -$

$2B = 30^\circ$

$\Rightarrow B = 15^\circ$. **Ans.**

Note : $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A + B) \neq \sin A + \sin B$.

TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

∵ We know complementary angles are pair of angles whose sum is 90°

Like 40°, 50° ; 60°, 30° ; 20°, 70° ; 15°, 75° ; etc,

Formulae :

$\sin (90^\circ - \theta) = \cos \theta$, $\cot (90^\circ - \theta) = \tan\theta$

$\cos (90^\circ - \theta) = \sin \theta$, $\sec (90^\circ - \theta) = \operatorname{cosec} \theta$

$\tan (90^\circ - \theta) = \cot \theta$, $\operatorname{cosec} (90^\circ - \theta) = \sec \theta$

Ex.32 Evaluate $\frac{\tan 65^\circ}{\cot 25^\circ}$. **[NCERT]**

Sol. ∵ $65^\circ + 25^\circ = 90^\circ$

$\therefore \frac{\tan 65^\circ}{\cot 25^\circ} = \frac{\tan (90^\circ - 25^\circ)}{\cot 25^\circ} = \frac{\cot 25^\circ}{\cot 25^\circ} = 1$

Ans.

Ex.33 Without using trigonometric tables, evaluate the following :

(i) $\frac{\cos 37^\circ}{\sin 53^\circ}$ (ii) $\frac{\sin 41^\circ}{\cos 49^\circ}$ (iii) $\frac{\sin 30^\circ 17'}{\cos 59^\circ 43'}$

Sol.(i) We have

$\frac{\cos 37^\circ}{\sin 53^\circ} = \frac{\cos(90^\circ - 53^\circ)}{\sin 53^\circ} = \frac{\sin 53^\circ}{\sin 53^\circ} = 1$

[∵ $\cos(90^\circ - \theta) = \sin \theta$]

(ii) We have,

$\frac{\sin 41^\circ}{\cos 49^\circ} = \frac{\sin(90^\circ - 49^\circ)}{\cos 49^\circ} = \frac{\cos 49^\circ}{\cos 49^\circ} = 1$

[∵ $\sin (90^\circ - \theta) = \cos \theta$]

(iii) We have,

$\frac{\sin 30^\circ 17'}{\cos 59^\circ 43'} = \frac{\sin(90^\circ - 59^\circ 43')}{\cos 59^\circ 43'} = \frac{\cos 59^\circ 43'}{\cos 59^\circ 43'} = 1.$

Ex.34 Without using trigonometric tables evaluate the following :

(i) $\sin^2 25^\circ + \sin^2 65^\circ$ (ii) $\cos^2 13^\circ - \sin^2 77^\circ$

Sol.(i) We have,

$\sin^2 25^\circ + \sin^2 65^\circ = \sin^2 (90^\circ - 65^\circ) + \sin^2 65^\circ$

$= \cos^2 65^\circ + \sin^2 65^\circ = 1$

[$\because \sin (90^\circ - \theta) = \cos \theta$]

(ii) We have,

$\cos^2 13^\circ - \sin^2 77^\circ = \cos^2(90^\circ - 77^\circ) - \sin^2 77^\circ$

$= \sin^2 77^\circ - \sin^2 77^\circ = 0$

[$\because \cos (90^\circ - \theta) = \sin \theta$]

Ex.35 Without using trigonometric tables, evaluate the following :

(i) $\dfrac{\cot 54^\circ}{\tan 36^\circ} + \dfrac{\tan 20^\circ}{\cot 70^\circ} - 2$

(ii) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

Sol.(i) We have,

$\dfrac{\cot 54^\circ}{\tan 36^\circ} + \dfrac{\tan 20^\circ}{\cot 70^\circ} - 2$

$= \dfrac{\cot(90^\circ - 36^\circ)}{\tan 36^\circ} + \dfrac{\tan 20^\circ}{\cot(90^\circ - 20^\circ)} - 2$

$= \dfrac{\tan 36^\circ}{\tan 36^\circ} + \dfrac{\tan 20^\circ}{\tan 20^\circ} - 2 = 1 + 1 - 2 = 0$

(ii) We have,

$\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

$= \sec(90^\circ - 40^\circ) \sin 40^\circ + \cos 40^\circ \operatorname{cosec}(90^\circ - 40^\circ)$

$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \sec 40^\circ$

$= \dfrac{\sin 40^\circ}{\sin 40^\circ} + \dfrac{\cos 40^\circ}{\cos 40^\circ} = 1 + 1 = 2$

Ex.36 Express each of the following in terms of trigonometric ratios of angles between 0º and 45º;

(i) $\operatorname{cosec} 69^\circ + \cot 69^\circ$

(ii) $\sin 81^\circ + \tan 81^\circ$

(iii) $\sin 72^\circ + \cot 72^\circ$

Sol.(i) We have,

$\operatorname{cosec} 69^\circ + \cot 69^\circ$

$= \operatorname{cosec} (90^\circ - 21^\circ) + \cot (90^\circ - 21^\circ)$

$= \sec 21^\circ + \tan 21^\circ$

[$\because \operatorname{cosec} (90^\circ - \theta) = \sec \theta$ and $\cot (90^\circ - \theta) = \tan \theta$]

(ii) We have,

$\sin 81^\circ + \tan 81^\circ$

$= \sin (90^\circ - 9^\circ) + \tan (90^\circ - 9^\circ)$

$= \cos 9^\circ + \cot 9^\circ$

[$\because \sin (90^\circ - \theta) = \cos \theta$ and $\tan (90^\circ - \theta) = \cot \theta$]

(iii) We have,

$\sin 72^\circ + \cot 72^\circ$

$= \sin (90^\circ - 18^\circ) + \cot (90^\circ - 18^\circ)$

$= \cos 18^\circ + \tan 18^\circ$

[$\because \sin (90^\circ - 18^\circ) = \cos 18^\circ$ and $\tan (90^\circ - 18^\circ) = \cot 18^\circ$]

Ex.37 Without using trigonometric tables, evaluate the following :

$$\frac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

Sol.

$$\frac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

$$= \frac{\sin^2 20^\circ + \sin^2 (90^\circ - 20^\circ)}{\cos^2 20^\circ + \cos^2 (90^\circ - 20^\circ)} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$$

$$= \frac{\sin^2 20^\circ + \cos^2 20^\circ}{\cos^2 20^\circ + \sin^2 20^\circ} + \frac{\cos\theta\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta\cos\theta}{\frac{\cos\theta}{\sin\theta}}$$

$$\left[\begin{aligned}\sin(90^\circ - \theta) &= \cos\theta \text{ and} \\ \cos(90^\circ - \theta) &= \sin\theta\end{aligned}\right]$$

$= \dfrac{1}{1} + \cos^2 \theta + \sin^2 \theta = 1 + 1 = 2$

Ex.38 If $\tan 2\theta = \cot (\theta + 6^\circ)$, where 2θ and $\theta + 6^\circ$ are acute angles, find the value of θ.

Sol. We have,

$\tan 2\theta = \cot (\theta + 6^\circ)$

$\Rightarrow \cot(90^\circ - 2\theta) = \cot (\theta + 6^\circ)$

$\Rightarrow 90^\circ - 2\theta = \theta + 6^\circ \Rightarrow 3\theta = 84^\circ$

$\Rightarrow \theta = 28^\circ$

Ex.39 If A, B, C are the interior angles of a triangle ABC, prove that $\tan \dfrac{B + C}{2} = \cot \dfrac{A}{2}$

Sol. In ΔABC, we have

$A + B + C = 180^\circ$

$\Rightarrow B + C = 180^\circ - A$

$\Rightarrow \dfrac{B + C}{2} = 90^\circ - \dfrac{A}{2}$

$\Rightarrow \tan\left(\frac{B+C}{2}\right) = \tan\left(90° - \frac{A}{2}\right)$

$\Rightarrow \tan\left(\frac{B+C}{2}\right) = \cot\frac{A}{2}$

Ex.40 If tan 2A = cot (A – 18°), where 2A is an acute angle, find the value of A. **[NCERT]**

Sol. tan 2A = cot (A – 18°)

cot (90° – 2A) = cot (A – 18°)

($\because$ cot (90° – θ) = tan θ)

90° – 2A = A – 18°

3A = 108°

A = 36° **Ans.**

Ex.41 If tan A = cot B, prove that A + B = 90°.

Sol. $\because$ tan A = cot B

tan A = tan (90° – B)

A = 90° – B

A + B = 90°. Proved

Ex.42 If A, B and C are interior angles of a triangle ABC, then show that

$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ **[NCERT]**

Sol. $\because$ A + B + C = 180° (a.s.p. of Δ)

B + C = 180° – A

$\left(\frac{B+C}{2}\right) = 90° - \frac{A}{2}$

$\sin\left(\frac{B+C}{2}\right) = \sin\left(90° - \frac{A}{2}\right)$

$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ Proved.

Ex.43 Express sin 67° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

Sol. $\because$ 23 = 90 – 67 & 15 = 90 – 75

$\therefore$ sin 67° + cos 75°

= sin (90 – 23)° + cos (90 – 15)°

= cos 23° + sin 15°. **Ans.**

➤ TRIGONOMETRIC IDENTITIES

(1) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (linear)

$$\left.\begin{array}{ll} (2) & \sin^2\theta + \cos^2\theta = 1 \\ (3) & 1 + \tan^2\theta = \sec^2\theta \\ (4) & 1 + \cot^2\theta = \text{cosec}^2\theta \end{array}\right]\text{squareidentites}$$

❖ EXAMPLES ❖

Ex.45 Prove the following trigonometric identities :

(i) $\frac{\sin\theta}{1-\cos\theta}$ = cosecθ + cot θ

(ii) $\frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta} = \frac{\sec\theta + 1}{\sec\theta - 1}$

Sol.(i) We have,

LHS = $\frac{\sin\theta}{(1-\cos\theta)} \times \frac{(1+\cos\theta)}{(1+\cos\theta)}$

[Multiplying numerator and denominator by (1 + cosθ)]

$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$

[$\because$ 1 – cos²θ = sin²θ]

$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$

= cosecθ + cotθ = RHS

$\left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta \text{ and } \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$

(ii) We have,

LHS = $\frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta}$

$= \frac{\frac{\sin\theta}{\cos\theta} + \sin\theta}{\frac{\sin\theta}{\cos\theta} - \sin\theta} = \frac{\sin\theta\left(\frac{1}{\cos\theta} + 1\right)}{\sin\theta\left(\frac{1}{\cos\theta} - 1\right)}$

$\frac{\frac{1}{\cos\theta} + 1}{\frac{1}{\cos\theta} - 1} = \frac{\sec\theta + 1}{\sec\theta - 1}$ = RHS

Ex.46 Prove the following identities :

(i) (sinθ + cosecθ)² + (cosθ + secθ)² = 7 + tan²θ + cot²θ

(ii) (sinθ + secθ)² + (cosθ + cosecθ)² = (1 + secθ cosecθ)²

(iii) sec⁴θ – sec²θ = tan⁴θ + tan²θ

Sol.(i) We have,

LHS = (sinθ + cosecθ)² + (cosθ + secθ)²

= (sin²θ + cosec²θ + 2sinθ cosecθ)

(cos²θ + sec²θ + 2cosθ secθ)

$= \left(\sin^2\theta + \text{cosec}^2\theta + 2\sin\theta.\frac{1}{\sin\theta}\right)$

$$+\left(\cos^2\theta+\sec^2\theta+2\cos\theta.\frac{1}{\cos\theta}\right)$$

$= (\sin^2\theta + \text{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2)$

$= \sin^2\theta + \cos^2\theta + \text{cosec}^2\,\theta + \sec^2\theta + 4$

$= 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4$

$[\because\ \text{cosec}^2\theta = 1 + \cot^2\theta,\ \sec^2\theta = 1 + \tan^2\theta]$

$= 7 + \tan^2\theta + \cot^2\theta$ = RHS.

(ii) We have,

LHS $= (\sin\theta + \sec\theta)^2 + (\cos\theta + \text{cosec}\theta)^2$

$$= \left(\sin\theta+\frac{1}{\cos\theta}\right)^2 + \left(\cos\theta+\frac{1}{\sin\theta}\right)^2$$

$$= \sin^2\theta + \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + \cos^2\theta + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}$$

$$= (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}\right) + 2\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$= (\sin^2\theta + \cos^2\theta) + \left(\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}\right) + \frac{2(\sin^2\theta+\cos^2\theta)}{\sin\theta\cos\theta}$$

$$= 1 + \frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin\theta\cos\theta}$$

$$= \left(1+\frac{1}{\sin\theta\cos\theta}\right)^2 = (1+\sec\theta\,\text{cosec}\theta)^2 = \text{RHS}$$

(iii) We have, LHS $= \sec^4\theta - \sec^2\theta$

$= \sec^2\theta\,(\sec^2\theta - 1) = (1 + \tan^2\theta)\,(1 + \tan^2\theta - 1)$

$[\because\ \sec^2\theta = 1 + \tan^2\theta]$

$= (1 + \tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta$ = RHS.

Ex.61 Express the ratios cos A, tan A and sec A in terms of sin A.

Sol. Since $\cos^2A + \sin^2A = 1$, therefore,

$\cos^2A = 1 - \sin^2A$, i.e., $\cos A = \pm\sqrt{1-\sin^2 A}$

This gives

$\cos A = \sqrt{1-\sin^2 A}$ (Why ?)

Hence,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}}\ \text{and}$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}}.$$

Ex.62 Prove that $\dfrac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \dfrac{1}{\sec\theta-\tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Sol. $$\text{LHS} = \frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{\tan\theta-1+\sec\theta}{\tan\theta+1-\sec\theta}$$

$$= \frac{(\tan\theta+\sec\theta)-1}{(\tan\theta-\sec\theta)+1}$$

$$= \frac{\{(\tan\theta+\sec\theta)-1\}(\tan\theta-\sec\theta)}{\{(\tan\theta-\sec\theta)+1\}(\tan\theta-\sec\theta)}$$

$$= \frac{(\tan^2\theta-\sec^2\theta)-(\tan\theta-\sec\theta)}{\{\tan\theta-\sec\theta+1\}(\tan\theta-\sec\theta)}$$

$$= \frac{-1-\tan\theta+\sec\theta}{(\tan\theta-\sec\theta+1)(\tan\theta-\sec\theta)}$$

$$= \frac{-1}{\tan\theta-\sec\theta} = \frac{1}{\sec\theta-\tan\theta},$$

which is the RHS of the identity, we are required to prove.